I am an algebraic geometer studying the geometric and combinatorial structures that arise from birational geometry and other fields such as the theory of polytopes and machine learning. I am currently a visiting assistant professor at the University of California, Riverside. I am currently collaborating with three with interdisciplinary teams of experts from the U. of British Columbia, UC Los Angeles, UC Riverside, UMass Amherst and WashU at St. Louis; two in geometry and one in the mathematics of machine learning.

The main focus of my research it to unravel the hidden structures of *negative curves* in surfaces known as *blow-ups of weighted projective planes*. These curves largely govern the geometry of such surfaces and, in particular, lie at the heart of determining whether they are *Mori dream spaces* (MDS). Knowing if a variety is a MDS is of special significance in the minimal model program, a central project that aims to classify projective algebraic varieties. This area has seen explosive growth recently, since the MDS property for many important varieties can be inferred from these spaces.

My second research focus is in the study of the complexity of *convolutional neural networks* (CNNs), which are central tools in the application of machine learning to image processing. *Max-pooling layers* (MPLs) are key components of these networks. From a mathematical point of view, MPLs are piecewise linear functions, and counting their number of linearity regions is an active area of research. This combinatorial problem is related to several other problems in mathematics, for example, finding the vertices of Minkowski sums of polytopes.

1 Discovering the hidden structure of negative curves

Mori dream spaces (MDS) are varieties of special significance in the minimal model program, a thriving project within algebraic geometry that aims to classify all complex projective varieties by constructing a model of any such variety which is as simple as possible, this is, a *minimal model.* Mori dream spaces possess many remarkable properties, e.g., if X is a MDS it can be fully reconstructed from its *Cox ring*, a ring encompassing all information about meromorphic functions on X. Varieties such as projective spaces, toric varieties, Grassmannians and flag varieties are all MDS.

A divisor D in a variety X is a bookkeeping device used to construct meromorphic functions on X having zeros and poles prescribed by D; formally, a divisor is a formal $\mathbb{Z}$-combination of codimension-1 subvarieties of X. The vector space of all such functions is denoted by $H^0(X, \mathcal{O}_X(D))$. The *Cox ring* of X is defined as:

$$\operatorname{Cox}(X) = \bigoplus_{D \text{ divisor}} H^0(X, \mathcal{O}_X(D))$$

This ring contains all the necessary information to find the minimal model of X. However, in practice, this is only feasible when the $\operatorname{Cox}(X)$ is finitely generated; when this is the case X is said to be a Mori dream space [17].

Blow-ups of weighted projective planes. Castravet-Tevelev [4] showed that the space parameterizing genus zero curves with n marked points, $\overline{\mathcal{M}}_{0,n}$, is not a MDS for $n \gg 1$. They reduce the problem to showing the blow-up of a weighted projective plane at a general point is not MDS and invoke known examples of such phenomenon [16]. WPPs are prototypical examples of toric varieties.

This problem has a long tradition in commutative algebra, where these Cox rings are known as symbolic Rees algebras of monomial curves. Moreover, the study of these blow-ups may offer a new direction to tackle Nagata's conjecture. Known results include that X is MDS if $\min\{a, b, c\} = 1, 2, 3, 4, 6$ [18, 24] or if $(a+b+c)^2 > abc$ [7]. The novelty of our approach lies in using toric techniques to study these blow-ups; the following deceptively simple result is an example of its effectiveness:

Theorem 1. [14] If $b \mid a + c$, then $X = \operatorname{Bl}_e \mathbb{P}(a, b, c)$ is a MDS.

Cutkosky showed that these blow-ups are MDS iff they contain two disjoint curves different from the exceptional divisor, C and D [7]. The self-intersection of one of these curves, C, is necessarily non-positive, so C is called a *negative curve.* A remarkable outcome of our research is that negative curves govern the MDS property for these varieties beyond what one would expect from this result.

By toric geometry, a negative curve C is the same as a polynomial $f \in \mathbb{C}[x, y]$ together with a rational triangle $\Delta \subseteq \mathbb{R}^2$ such that f is *supported in* Δ, i.e., the Newton polygon of f is contained in Δ. Then, the self-intersection of the curve C is $C \cdot C = 2\operatorname{Area}(\Delta) - m^2$, where m is the order of vanishing of f at $e = (1, 1)$. Our strategy then consists on constructing polynomials $f \in \mathbb{C}[x, y]$

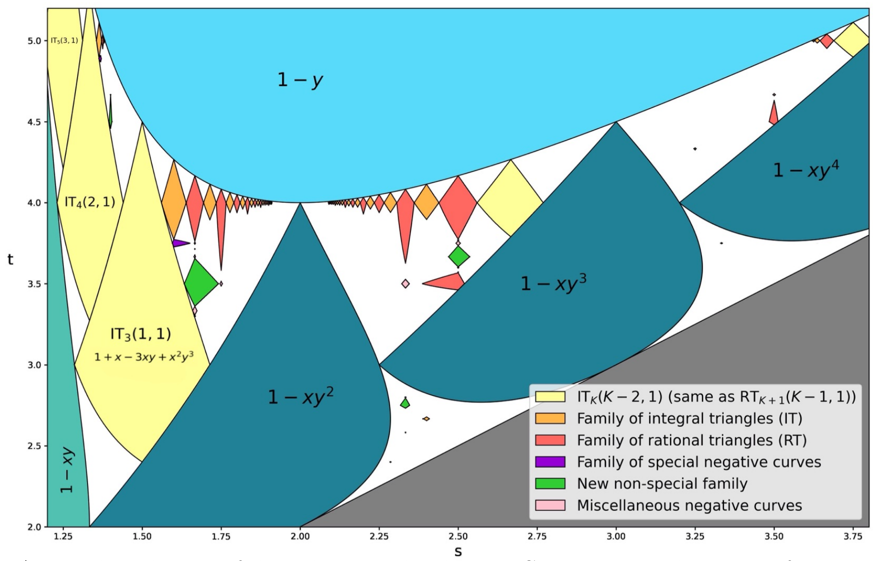


Figure 1: A novel and powerful tool to study the MDS property: the map of negative curves.

vanishing to order m at e and supported in *some* triangle Δ with area $\mathrm{Area}(\Delta) \le \frac{m^2}{2}$. This approach can be successfully used for REUs, see for example [6, 21, 1].

As part of our program we have constructed the first four known infinite families of negative curves [13, 14] and used them to find MDS and non-MDS examples [11, 12, 13, 14]. Figure 1 shows a parameter space of triangles and the negative curves they support. WLOG assume the triangles have a horizontal base. A point (s, t) in this space corresponds to a triangle with left-hand side slope s and right-hand side slope t. Each colored region corresponds to triangles Δ for which a fixed polynomial $f(x, y)$ defines a negative curve in the blow-up of the WPP defined by Δ.

Once a negative curve C in X is known, it remains to determine whether there is a curve D disjoint from it. In [11] we use positive characteristic methods to study the problem. More recently, we show that the MDS property is better understood "collectively" by studying Figure 1 [14]. Indeed, it is often the case that polynomials defining negative curves in adjacent regions of the map define disjoint curves in the interior of the regions, see Figure 2a. In [14] we show that this explains and expands most (if not all) of the previously known results in the area.

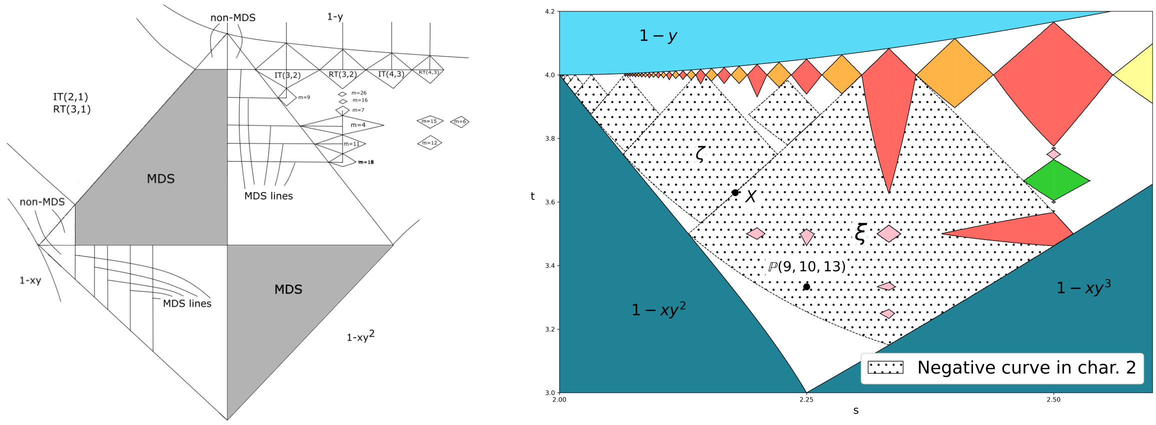
Finally, a natural question is whether every such variety contains a negative curve. In [15] we answer this question in the negative. Once again, the key is to explore Figure 1. However, we consider polynomials in characteristic 2. Over prime characteristic new negative curves emerge and previously known ones are supported in new triangles, see Figure 2b. In [15] we identify the cohomological obstruction that prevents a curve in char. 2 from lifting to a negative curve in char. 0. Then, we exploit the discrepancy between the geography of negative curves in both characteristics to find suitable candidates where the lifting is obstructed.

Theorem 2. [15] There are infinitely many triangles Δ such that $\mathrm{Bl}_e X_\Delta$ does not contain a negative curve.

Any X_Δ defined by a triangle is a quotient of $\mathbb{P}^2$ by a finite abelian group. This set includes all WPPs, however, the examples in the theorem are not of this kind. The non-existence of negative curves for a blow-up $\mathrm{Bl}_e \mathbb{P}(a, b, c)$ would prove Nagata's conjecture [23] for abc points and give an example of a surface with irrational Seshadri constant [3].

Future directions Many open questions are within reach of our methods:

1. The toric varieties in [15] are not WPPs. It remains to find examples where $\mathrm{Bl}_e \mathbb{P}(a, b, c)$ does not contain a negative curve. A known candidate is $\mathrm{Bl}_e \mathbb{P}(9, 10, 13)$ [20]. The obstruction theory we developed in [15] remains valid for these cases, but other difficulties need to be addressed.
2. Finding a non-MDS where $\min\{a, b, c\} = 5$ is an open problem. In [14] we explain the reason for this and conjecture there are examples where it is not a MDS. In fact, we conjecture there are examples not even containing a negative curve.
3. The geometric ideas presented in [14] recover and extend most of the previously known results in the area. However, the connection of our methods with those used by the commutative algebra



(a) The map reveals that the MDS property varies (almost) continuously in the parameter space of triangles.

(b) The map in char. 2 allowed us to find spaces with no negative curve!

Figure 2: The map of negative curves explains how the MDS property varies in families of blow-ups.

community remains to be explored. The tools we have developed can explain many of the known algebraic constructions and proofs. Currently, we are actively discussing the problem with algebraists from UNL. For example, Professor Alexandra Seceleanu pointed out to me that finite generation for blow-ups of WPPs at more than one point can be used to study finite generation problems for cases of interest in commutative algebra.

2 Understanding the complexity of neural networks

As a member of the AIMs Latinx Mathematicians Research Community (LMRC) I am involved in a project aiming to understand the complexity of convolutional neural networks; these are central tools in the application of machine learning to text, image and audio recognition. Despite their major success, machine learning algorithms have historically operated mostly as a black boxes. Their next stage of optimization requires a deeper understanding of the mathematics behind them. The following project is part of this stage of development.

Specifically, I study the combinatorics of max-pooling layers (MPLs), key components of these networks. From a mathematical point of view, MPLs are piecewise-linear functions; my research is concerned with counting their number of linearity regions [9], which is interpreted as a measure of complexity of the layer. There is a large undergoing effort to understand these networks [25, 22].

Concretely, an MPL is a function $f : \mathbb{R}^N \to \mathbb{R}^m$ of the form $(f(x))_a = \max_{i\in\lambda_a}\{x_i\}$, for subsets $\lambda_1, \dots, \lambda_m \subseteq \{0, 1, \dots, N\}$. The number of linearity regions of f is the same as the number of vertices of the Minkowski sum $P = \sum \Delta_{\lambda_a}$, where $\Delta_{\lambda_a} = \mathrm{conv}\{e_i : i \in \lambda_a\}$ and $\{e_i\}$ is the standard basis [19].

The subsets λ_a are determined by the geometry of the input array, see Figure 3 for an example of a 1-dimensional input layer. Moreover, these geometries can often be considered in families, for example, we solve the enumeration problem for large classes of 1D input layers (Theorem 3) and 2D input layers of size $3 \times N$ for $N > 0$ (Theorem 4). We encode these counts in generating functions.

Let us describe our results for 1-dimensional MPLs. Given positive integers n, k, s, let $P_{n,k,s}$ denote the polytope given by the Minkowski sum of the n simplices $\Delta_{\{si,si+1,\dots,si+k-1\}}$ for $i = 0, \dots, n-1$. The number s is known as the *stride* of the MPL. Let $b_n^{(k,s)}$ denote the number of vertices of $P_{n,k,s}$, which is also equal to the number of linearity regions corresponding MPL, see Figure 3.

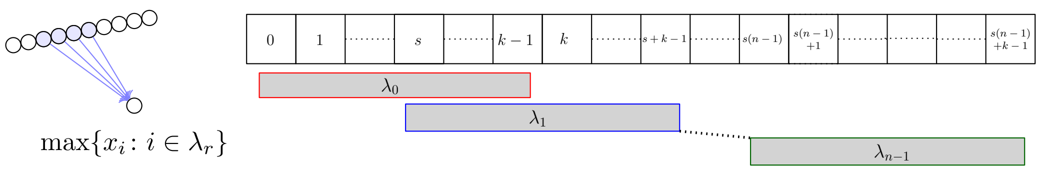


Figure 3: Showing the structure of 1-dimensional max-pooling layer and its mathematical counterpart.

We prove that the generating functions of the sequences $(b_n^{(k,s)})_{n\geq 1}$ are *rational* using the *transfer-matrix method* from enumerative combinatorics. Moreover, we give closed forms for these generating functions for the cases of *large strides*, i.e. $\lceil k/2 \rceil \leq s \leq k-2$, and *proportional strides*, i.e. $k = s(r+1)$.

Theorem 3. [9] Fix positive integers k and s.

(a) If $s \in \{\lceil k/2 \rceil, \ldots, k-2\}$, then the generating function of $(b_n^{(k,s)})_{n\geq 1}$ is

$$1 + \sum_{n\geq 1} b_n^{(k,s)} x^n = \frac{1}{1 - kx + (k-s)(k-s-1)x^2}.$$

(b) If $k = s(r+1)$, then the generating function of $(b_n^{(k,s)})_{n\geq 1}$ is

$$1 + \sum_{n\geq 1} b_n^{(k,s)} x^n = \frac{1 + (rs - s - 2)x - (rs-1)x^2 + sx^{r+1}}{1 - 2(s+1)x + (s+1)^2x^2 + sx^{r+1} - s^2(r+1)x^{r+2} + s(rs-1)x^{r+3}}.$$

Already the 2D case is significantly harder. Consider $\mathbb{R}^{3\times N} \cong \mathbb{R}^{3N}$ with basis $\{e_{i,j} \mid 0 \leq i \leq 2,\ 0 \leq j \leq N-1\}$, and let Q_n be the Minkowski sum of the $2(n-1)$ simplices $\Delta_{i,j} = \operatorname{conv}\{e_{i,j}, e_{i,j+1}, e_{i+1,j}, e_{i+1,j+1}\}$. Let V_n be the number of vertices of Q_n. Then,

Theorem 4. [9] The number V_n of vertices of the polytope Q_n is given by the generating function

$$x \ + \ \sum_{n\geq 2} V_n\, x^n \ = \ \frac{x + x^2 - x^3}{1 - 13x + 31x^2 - 20x^3 + 4x^4}.$$

From an algebro-geometric perspective it is possible to construct the toric variety X_P corresponding to the fan $\mathcal{N}(P)$. Then, enumerating the vertices of P is equivalent to computing the topological Euler characteristic $\chi(X_P)$ and Theorems 3 and 4 are also geometric results about toric varieties.

Future directions. The nature of this problem lends itself to a variety of interesting new directions:

1. Increasing the size or dimension of the input layers quickly results in significantly harder problems. However, it is not clear what the turning point is. Currently I am in a collaboration involving 3 undergraduate students to solve the 3D case for input layers of sizes $2 \times 3 \times N$.
2. Enumeration of higher dimensional faces of these polytopes is within reach. In [9] we also give a count of facets. I expect our techniques can be generalized to count edges too.
3. The composition of two of these max-pooling layers is once again a PWL function. Together with the same team of collaborators from [9] we are actively working on this enumeration problem. The collaboration has been awarded with an AIM SQuaREs to collaborate in person at the AIM headquarters for a week in August 2023.

3 Work in progress: Exploring higher dimensions

Let $U_{d,n}$ be the moduli space of n distinct labelled points in $\mathbb{C}^d$ up to translations and scaling. In [5], the authors construct a compactification of $U_{d,n}$ called $T_{d,n}$ possessing very desirable geometric and modular properties. Moreover, in the one dimensional case $d = 1$, we have $T_{1,n} = \overline{\mathcal{M}}_{0,n}$. In collaboration with P. Gallardo, J.L. González and E. Routis, I am working on determining when is $T_{d,n}$ a *weak Fano* variety and, when it is, to describe its cones of divisors. This falls within my goal to better understand the MDS property, since every weak Fano variety is a MDS.

It is well know that for $d = 1$ the $\overline{\mathcal{M}}_{0,4}$, $\overline{\mathcal{M}}_{0,5}$, and $\overline{\mathcal{M}}_{0,6}$ [2] are the only weak Fano cases. In general, $T_{d,2} \cong \mathbb{P}^{d-1}$, so the next natural case of study is $T_{d,3}$.

Our strategy relies on the fact that $T_{d,n}$ can be constructed as a sequence of smooth blow-ups of $\mathbb{P}^{d(n-1)-1}$ [10]. This procedure generalizes Kapranov's construction of $\overline{\mathcal{M}}_{0,n}$ and allows for a explicit description of $T_{d,n}$. In particular, $T_{d,3}$ is the blow-up of $\mathbb{P}^{2d-1}$ at the three disjoint linear subspaces.

A variety is said to be Weak Fano if its *anticanonical divisor* is *nef* and *big*. Within this context, the case $d = 1$ was solved in [8]. The general case is accessible as well. To prove that $T_{d,3}$ is weak Fano it is enough to show that, given any point $p \in \mathbb{P}^{2d-1}$, not in any of the three linear spaces, there exists a homogeneous polynomial of degree $2d+2$ in vanishing along the three linear subspaces to order d, but not passing through p. Once a divisor is known to be nef, bigness can be directly checked by means of standard intersection theory. This allows for a hands-on combinatorial approach that we are currently pursuing. Preliminary results show this is only true whenever $d \leq 9$.

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